Journal of Statistical Physics, Vol. 29, No. 2, 1982

The Spectral Decomposition for a Class of Linear Transport Operators

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Received April 5, 1982

The spectral decomposition theorem is proved for linear transport operators with separable scattering kernels and energy-dependent collision frequency in both one- and three-dimensional geometries.

KEY WORDS: Linear transport (Boltzmann) operator; two-dimensional spectrum; spectral decomposition; generalized eigenfunction expansion.

1. INTRODUCTION

In a previous paper[1] a first proof was given within a Hilbert space setting, for the spectral decomposition of a linear monoenergetic transport operator arising from time-dependent problems. The initial proof including both an infinite geometry and a slab geometry with periodic boundary conditions, was later extended for the same model (the Lorentz gas) considered in semi-infinite geometry with a specularly reflecting boundary.[2] In all these cases a Fourier analysis of the problem is relevant and one can replace the study of the original transport operator by the study of its Fourier components (or "reduced" transport operators). The essential features of the Lorentz model (such as the constancy of the collision frequency and the finite range of the velocity variable) combined with the Fourier decomposition lead to an enormous simplification, allowing the proof of the spectral decomposition theorem.[1,2] The aim of this paper is to extend the method

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0022-4715/82/1000-0363$03.00/0

applied in the previous work to the case of an energy-dependent collision frequency, a noncompact domain for the velocity variable and a two-dimensional structure for the spectrum of the transport operator. More precisely, we shall effectively construct, as before, the intertwining operators which realize the similarity between the reduced transport operators and some normal operators, for which the spectral decomposition is given by the general theory.

2. THE MODELS AND THE SETTING OF THE PROBLEM

Let us consider the action of the linear transport operator

$$(Af)(\mathbf{r},\mathbf{v}) = -\mathbf{v}\nabla_{\mathbf{r}} f(\mathbf{r},\mathbf{v}) - \xi(v) f(\mathbf{r},\mathbf{v}) + \beta \int_{\mathbb{R}^3} \xi(v)\xi(v') M(v) f(\mathbf{r},\mathbf{v}')\, d\mathbf{v}' \quad (1)$$

considered in a certain Hilbert space, where $v = |\mathbf{v}|$, $M(v) = e^{-v^2}$, the Maxwell flux, $\xi(v)$ is the collision frequency, and $\beta^{-1} = \int \xi(v) M(v)\, dv$. In the following, $\xi(v)$ might be a rather general function with the property $\inf_v \xi(v) = \xi(0) \equiv \lambda^* > 0$.

A model of the kind shown in Eq. (1) arises from studying shear flows with a velocity dependent Krook model (see Ref. 3 and references therein). The same model was also proposed by Corngold *et al.*[(4)] in neutron tranport. In the particular case when $\xi(v)$ is a constant, we obtain the linearized BGK model for shear flows.[(3)] The scattering operator given in (1) is nonsymmetric in the Hilbert space $L_2(R^3 \times R^3)$ but becomes symmetric (even self-adjoint) if we introduce $M(v)$ as a weight function, i.e., if we consider its action in $L_2(R^3 \times R^3; M(v)\, d\mathbf{r}\, d\mathbf{v})$. Alternatively, we can symmetrize the scattering operator by changing the function f upon which it acts into $g(\mathbf{v}) = f(\mathbf{v})[M(\mathbf{v})]^{-1/2}$. We shall look for solutions independent of the polar angle of $\mathbf{v}$ and shall consider only functions of $\mathbf{r}$, v, μ, where μ is the cosine of the angle between $\mathbf{v}$ and the polar axis in the velocity space.

Accordingly, we shall consider the symmetrized form of A acting on functions belonging to the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^3 \times \mathbb{R}_+ \times [-1, 1])$

$$(Af)(\mathbf{r}, v, \mu) = -\mathbf{v}\nabla_{\mathbf{r}} f(\mathbf{r}, v, \mu) - \xi(v) f(\mathbf{r}, v, \mu) + 2\pi\beta \times \int_0^\infty \int_{-1}^1 \xi(v)\xi(v') \left[M(v) M(v') \right]^{1/2} v v' f(\mathbf{r}, v', \mu')\, dv'\, d\mu' \quad (2)$$

A Fourier transformation in the spatial coordinate $\mathbf{r}$ allows us to identify the transport operator A as the orthogonal integral

$$A = \int_{\oplus} A_k\, dk \quad (3)$$

where each reduced transport operator A_k acting in $\mathcal{L} \equiv L_2(\mathbb{R}_+ \times [-1, 1])$ is given by

$$A_k = ikv\mu - \xi(v) + J$$

$$(Jf)(k, v, \mu) \equiv 2\pi\beta \int_0^\infty \int_{-1}^1 \xi\xi'(MM')^{1/2} vv' f(k, v', \mu')\, dv'\, d\mu' \tag{4}$$

$$M \equiv M(v), \qquad M' \equiv M(v'), \qquad \xi \equiv \xi(v), \qquad \xi' \equiv \xi(v')$$

In order to prove that A is spectral in the sense of Dunford[5] we have to find a norm-bounded, strongly additive function $E: B \to \mathcal{B}(\mathcal{H})$ defined on B (the Borel sets of the complex plane, $\mathbb{C}$) with values in the bounded operators on $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$, such that $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$,

$$E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2) - E(\Delta_1 \cap \Delta_2), \quad E(\mathbb{C}) = I_{\mathcal{H}}, \quad \forall \Delta_1, \Delta_2 \in B$$

and such that the operator A can be diagonalized:

$$A = \int_{\mathbb{C}} \lambda E(d\lambda) \tag{5}$$

In view of the orthogonal decomposition (3), this has to be proved for every operator A_k, which for $k \neq 0$ are unbounded non-normal operators on L. [A_0 is a self-adjoint operator, bounded or unbounded, depending on the range of $\xi(v)$.] A_k can be viewed as the perturbation of the normal operator $ikv - \xi(v)$ by the self-adjoint operator of rank 1, J. Such problems have been extensively studied in the literature in a quite general framework. See, for instance, Friedrichs,[6] and for further developments Kato,[7] Dunford and Schwartz,[5] and references therein.

At least two approaches are available. The first one is to try a generalized eigenfunction expansion for A_k, for which the natural framework consists of rigged Hilbert spaces (Gel'fand triplets).[8] The second approach, invented by Friedrichs[6] consists of applying a scattering formalism, i.e., of constructing (nonunitary) bounded wave operators which realize the intertwining (similarity) of the perturbed operator A_k with a normal operator $\tilde{A}_k$ for which the spectral measure is simply available. Apart from its elegance, the formal advantage of this method is that only Hilbert space operators are involved. The equivalence of the two methods is easily established and can be inferred from the fact that the formal kernels of the wave operators realizing the intertwining are nothing but the generalized eigenfunctions of the perturbed operator.

In principle, the cases at hand are essentially covered by Friedrichs' result except that some new details are added in order to handle the more complicated case of a non-normal operator with a two-dimensional spectrum. However, in view of the interest presented by the method for linear transport theory, we shall present some details of Friedrichs' argument.

For convenience, instead of A_k we shall study the operator

$$B_k = v\mu + \frac{\xi i}{k} + \frac{2\pi\beta}{k}\int_0^{\infty}\int_{-1}^{1} \xi\xi'(MM')^{1/2} vv'\, dv'\, d\mu' \tag{6}$$

obtained from A_k by dividing it with ik ($k \neq 0$).

Therefore, we try to find a normal operator $\tilde{B}_k$ acting in a Hilbert space $\tilde{\mathfrak{L}}$ (to be determined too) and two bounded operators $\Omega_k^+ : \tilde{\mathfrak{L}} \to \mathfrak{L}$, $\Omega_k^- : \mathfrak{L} \to \tilde{\mathfrak{L}}$ with the following properties:

(a) $\Omega_k^+ \Omega_k^- = I_{\mathfrak{L}}$ ("completeness") (7)

(b) $\Omega_k^- \Omega_k^+ = I_{\tilde{\mathfrak{L}}}$ ("orthogonality") (8)

(c) $B_k \Omega_k^+ = \Omega_k^+ \tilde{B}_k, \qquad \Omega_k^- B_k = \tilde{B}_k \Omega_k^-$ (9)

Indeed, if this is done, then the spectral measure of A_k ($k \neq 0$) will be written as

$$E_k(\cdot) = ik\Omega_k^+ \tilde{E}_k(\cdot)\Omega_k^- \tag{10}$$

where $\tilde{E}_k(\cdot)$ is the (known) spectral measure of the normal operator $\tilde{B}_k$.

The spectrum of B_k is (for the proof see, for instance, Corngold[9]) the (1) essential spectrum at the left of the curve $\mathcal{C}$ described parametrically by

$$\begin{cases} \operatorname{Re} z = \pm v, \\ \operatorname{Im} z = \dfrac{\xi(v)}{k}, \end{cases} \qquad v \in \mathbb{R}_+ \tag{11}$$

(2) the discrete spectrum given by the dispersion equation

$$1 = \frac{2\pi\beta}{ki}\int_0^{\infty}\int_{-1}^{1} \frac{\xi^2 M v^2\, dv\, d\mu}{\lambda_k - v - \xi i/k} \tag{12}$$

These eigenvalues are purely imaginary and nondegenerate. For $k > 0$, $\lambda_k \in (0, \lambda^*/ik)$, and for $k < 0$, $\lambda_k \in (0, i(\lambda^*/k))$.

The corresponding normalized eigenvector is

$$f_k = C_k \frac{\xi M^{1/2} v}{\lambda_k - v - \xi i/k} \tag{13}$$

where C_k is given by the condition

$$\int_0^{\infty}\int_{-1}^{1} f_k^2\, dv\, d\mu = 1$$

There exists a critical value k^*, $0 < k^* < \infty$, such that for $|k| > k^*$ no discrete eigenvalues occur.

We shall distinguish therefore two cases:

(i) $|k| > k^*$, when B_k has no discrete eigenvalues; and

(ii) $|k| < k^*$, when B_k has the nondegenerated eigenvalue λ_k given by (12).

In case (i) we take $\tilde{\mathfrak{L}} \equiv \mathfrak{L}$ and $\tilde{B}_k \equiv B^0$

$$(B^0 n)(v, \mu) = vn(v, \mu) + (\xi/ik)n(v, \mu) \tag{14}$$

Conditions (a), (b), and (c) take the form

(a′) $$U_k^+ U_k^- = I_{\mathfrak{L}} \tag{7′}$$

(b′) $$U_k^- U_k^+ = I_{\mathfrak{L}} \tag{8′}$$

(c′) $$B_k U_k^+ = U_k^+ B^0, \qquad U_k^- B_k = B^0 U_k^- \tag{9′}$$

In case (ii), $\tilde{\mathfrak{L}}$ will be the orthogonal sum $\tilde{\mathfrak{L}} \equiv \mathfrak{L} \oplus \mathbb{C}$ (with vectors written as $\binom{f}{\eta}$, $f \in \mathfrak{L}$, $\eta \in \mathbb{C}$) and $\tilde{B}_k = \begin{pmatrix} B^0 & 0 \\ 0 & \lambda_k \end{pmatrix}$ with B^0 defined in (14). In this matrix notation the action of the wave operators Ω_k^+, Ω_k^- is given by

$$\begin{aligned} \Omega_k^+ \begin{pmatrix} f \\ \eta \end{pmatrix} &= (U_k^+ f_k) \begin{pmatrix} f \\ \eta \end{pmatrix} = U_k^+ f + \eta f_k, \qquad \forall f \in \mathfrak{L}, \quad \eta \in \mathbb{C} \\ \Omega_k^- f &= \begin{bmatrix} U_k^- \\ (\bar{f}_k, \cdot) \end{bmatrix} f = \begin{bmatrix} U_k^- f \\ (\bar{f}_k, f) \end{bmatrix}, \qquad \forall f \in \mathfrak{L} \end{aligned} \tag{15}$$

and conditions (a), (b), and (c) read now as

(a″) $$U_k^+ U_k^- = I_{\mathfrak{L}} - f_k(\bar{f}_k, \cdot) \tag{7″}$$

(b″) $$U_k^- U_k^+ = I_{\mathfrak{L}} \tag{8″}$$

(c″) $$B_k U_k^+ = U_k^+ B^0, \qquad U_k^- B_k = B^0 U_k^- \tag{9″}$$

As (c′) and (c″) look identical, we shall use them for determining $U_k^{\pm}$ simultaneously for both cases. We shall then verify (a′), (b′) and (a″), (b″) for cases (i) and (ii), respectively.

3. CONSTRUCTION OF THE WAVE OPERATORS

Writing $B_k = B^0 + (1/ik)J$, we translate conditions (c′), (c″) into

$$[B^0, U_k^+] = -\frac{1}{ik} J U_k^+ \equiv -\alpha\nu R_k^+$$

$$[B^0, U_k^-] = \frac{1}{ik} U_k^- J \equiv \alpha\nu R_k^-$$

where $\alpha \equiv 2\pi\beta/ik$, $\nu \equiv \xi M^{1/2} v$ and R_k^+ and R_k^- will be shown to be degenerate integral operators of rank 1, namely,

$$(R_k^+ n)(v, \mu) = \int_0^\infty \int_{-1}^1 r_k^+ (v', \mu') n(v', \mu')\, dv'\, d\mu' \tag{16}$$

$$(R_k^- n)(v, \mu) = r_k^- (v, \mu) \int_0^\infty \int_{-1}^1 n(v', \mu')\, dv'\, d\mu' \tag{17}$$

Supposing that $r_k^\pm$ are known, we have to find a bounded operator Z satisfying the operational equation:

$$[B^0, Z] = R \tag{18}$$

which actually has an infinity of solutions. We shall show that Eq. (18) has a bounded solution whenever R has a separable kernel of the form $r_1(v, \mu) r_2(v', \mu')$, $r_{1,2} \in L_\infty(R_+ \times [-1, 1])$, by effectively constructing one solution $Z = \Gamma(R)$ depending homogeneously on R.

$$\Gamma(R) = \hat{r}_1 \cdot (v\mu + i\xi/k - v'\mu' - i\xi'/k) \cdot \hat{r}_2 \equiv \hat{r}_1 (z - z')^{-1} \cdot \hat{r}_2 \tag{19}$$

where $\hat{r}_1, \hat{r}_2$ are the multiplicative operators by $r_1(v, \mu), r_2(v', \mu')$, respectively, and

$$z \equiv v\mu + i\xi/k \tag{20}$$

Therefore we shall write

$$U_k^+ = I_{\mathfrak{L}} + \Gamma(R_k^+) \tag{21}$$

$$U_k^- = I_{\mathfrak{L}} + \Gamma(R_k^-) \tag{22}$$

where we require that in the limit of null perturbation ($J = 0$) the equivalence be realized by the identity operator $I_{\mathfrak{L}}$, and where $R_k^\pm$ are to be determined from the consistency relations

$$R_k^+ = -\frac{i}{k\alpha\nu} J [I_{\mathfrak{L}} + \Gamma(R_k^+)]$$

$$R_k^- = \frac{i}{k\alpha\nu} [I_{\mathfrak{L}} + \Gamma(R_k^-)] J$$

In terms of formal kernels, Eq. (21) reads

$$u_k^+ (v, \mu, v', \mu') = \frac{1}{2\pi v^2} \delta(v - v') \delta(\mu - \mu') - \frac{\alpha\nu}{z - z'} r_k^+ (v, \mu, v', \mu') \tag{23}$$

where

$$r_k^+ (v, \mu, v', \mu') = \int_0^\infty \int_{-1}^1 \nu'' u_k^+ (v'', \mu'', v', \mu')\, dv''\, d\mu''$$
$$\nu'' \equiv \nu(v'') \tag{24}$$

By multiplying with and integrating over $dv\, d\mu$ we obtain from (22) the expression for $r_k^+(v, \mu, v', \mu')$ (which actually does not depend on v, μ):

$$r_k^+(v, \mu, v', \mu') = \frac{\nu'}{z - z'}(1 + \alpha\gamma')^{-1} \tag{25}$$

$$\gamma' \equiv \gamma(z') = \int_0^\infty \int_{-1}^1 \frac{\nu^2\, dv\, d\mu}{z - z'} \tag{26}$$

This leads finally to

$$u_k^+(v, \mu, v', \mu') = \frac{1}{2\pi v^2}\delta(v - v')\delta(\mu - \mu') - \frac{\alpha\nu\nu'}{z - z'}(1 + \alpha\gamma')^{-1} \tag{27}$$

Similarly, from Eq. (22) one gets

$$u_k^-(v, \mu, v', \mu') = \frac{1}{2\pi v^2}\delta(v - v')\delta(\mu - \mu') + \frac{\alpha\nu\nu'}{z - z'}(1 + \alpha\gamma)^{-1} \tag{28}$$

One should emphasize that the meaning of these expressions is that of formal kernels of nonintegral operators acting in $\mathfrak{L}$. For instance $(1/2\pi v^2)$ $\delta(v - v')\delta(\mu - \mu')$ has to be seen as the formal kernel of the identity operator $I_\mathfrak{L}$ and not as a distribution (as the functions considered here are vectors in L_2 and their value in a point is not always meaningful).

Now, by using the expression (27) and (28), one has to verify the relations (a′), (b′), (a″), (b″). Actually (b′) and (b″) are the same; to prove them one has to show that

$$U_k^- U_k^+ = I_\mathfrak{L} + \Gamma(R_k^-)\Gamma(R_k^+) + \Gamma(R_k^-) + \Gamma(R_k^+) = I_\mathfrak{L}$$

i.e., to show that $\Gamma(R_k^-)\Gamma(R_k^+) + \Gamma(R_k^-) + \Gamma(R_k^+) = 0$. Indeed, by computing the kernel of $\Gamma(R_k^-)\Gamma(R_k^+)$, one has

$$\begin{aligned}
&\int_0^\infty \int_{-1}^1 \alpha^2 \frac{\nu\nu''}{z'' - z}(1 + \alpha\gamma)^{-1}(1 + \alpha\gamma')^{-1}\frac{\nu''\nu'}{z'' - z'}\, dv''\, d\mu'' \\
&\quad = \alpha^2\nu\nu'(1 + \alpha\gamma)^{-1}(1 + \alpha\gamma')^{-1} \\
&\qquad \times \int_0^\infty \int_{-1}^1 \frac{v''^2}{z - z'}\left(\frac{1}{z'' - z} - \frac{1}{z'' - z'}\right) dv''\, d\mu'' \\
&\quad = \frac{\alpha\nu\nu'}{z - z'}(1 + \alpha\gamma)^{-1}(1 + \alpha\gamma')^{-1}(1 - \alpha\gamma' - 1 + \alpha\gamma) \\
&\quad = -\frac{\alpha\nu\nu'}{z - z'}(1 + \alpha\gamma')^{-1} - \alpha\frac{\nu\nu'}{z - z'}(1 + \alpha\gamma)^{-1}
\end{aligned} \tag{29}$$

which is precisely the kernel of the operator $-\Gamma(R_k^+) - \Gamma(R_k^-)$.

We shall now compute $U_k^+ U_k^-$ in the case (a″), when a discrete eigenvalue is present. The case (a′) is, of course, simpler. The formal kernel of $\Gamma(R_k^+)\Gamma(R_k^-) + \Gamma(R_k^-) + \Gamma(R_k^+)$ is

$$\int_0^\infty \int_{-1}^1 \alpha^2 \frac{\nu\nu''}{z - z''} (1 + \alpha\gamma'')^{-1} \frac{\nu'\nu''}{z' - z''} (1 + \alpha\gamma'')^{-1} dv''\, d\mu''$$
$$- \alpha \frac{\nu\nu'}{z' - z} (1 + \alpha\gamma)^{-1} - \alpha \frac{\nu\nu'}{z - z'} (1 + \alpha\gamma')^{-1}$$
$$= \alpha^2 \frac{\nu\nu'}{z' - z} \int_0^\infty \int_{-1}^1 \frac{\nu''^2}{z - z''} (1 + \alpha\gamma'')^{-2} dv''\, d\mu''$$
$$+ \alpha \frac{\nu\nu'}{z - z'} \int_0^\infty \int_{-1}^1 \frac{\nu''^2}{z' - z''} (1 + \alpha\gamma'')^{-2} dv''\, d\mu''$$
$$- \alpha \frac{\nu\nu'}{z' - z} (1 + \alpha\gamma)^{-1} - \alpha \frac{\nu\nu'}{z - z'} (1 + \alpha\gamma')^{-1} \tag{30}$$

The technical problem is now to estimate the integral terms in (30). To further this aim we shall use some results of the theory of generalized analytic functions[10] (see Appendix). Thus, one rewrites (30) in the form

$$\alpha \frac{\nu\nu'}{z' - z} \left[\frac{1}{\pi} \int_0^\infty \int_{-1}^1 \frac{\rho'' dv''\, d\mu''}{z - z''} \frac{\partial}{\partial \bar{z}''} (1 + \alpha\gamma'')^{-1} - (1 + \alpha\gamma)^{-1} \right]$$
$$+ \alpha \frac{\nu\nu'}{z - z'} \left[\frac{1}{\pi} \int_0^\infty \int_{-1}^1 \frac{\rho'' dv''\, d\mu''}{z' - z''} \frac{\partial}{\partial \bar{z}''} (1 + \alpha\gamma'')^{-1} - (1 + \alpha\gamma')^{-1} \right]$$
$$= \alpha \frac{\nu\nu'}{z' - z} \left\{ \frac{1}{\pi} \int_G\!\!\int \frac{\partial}{\partial \bar{z}''} \left[\frac{1}{z - z''} (1 + \alpha\gamma'')^{-1} \right] dx''\, dy'' \right.$$
$$\left. + \frac{1}{\pi} \int_G\!\!\int \pi\delta(z - z'')(1 + \alpha\gamma'')^{-1} dx''\, dy'' - (1 + \alpha\gamma)^{-1} \right\}$$
$$+ \alpha \frac{\nu\nu'}{z - z'} \left\{ \frac{1}{\pi} \int_G\!\!\int \frac{\partial}{\partial \bar{z}''} \left[\frac{1}{z' - z''} (1 + \alpha\gamma'')^{-1} \right] dx''\, dy'' \right.$$
$$\left. + \frac{1}{\pi} \int_G\!\!\int \pi\delta(z' - z'')(1 + \alpha\gamma'')^{-1} dx''\, dy'' - (1 + \alpha\gamma')^{-1} \right\}$$
$$= \alpha \frac{\nu\nu'}{z' - z} \frac{1}{\pi} \int_G\!\!\int \frac{\partial}{\partial \bar{z}''} \left[\left(\frac{1}{z - z''} - \frac{1}{z' - z''} \right) (1 + \alpha\gamma'')^{-1} \right] dx''\, dy'' \tag{31}$$

where ρ is defined by Eq. (A5) of the Appendix, G is the two-dimensional region covered by the essential spectrum, and where the relation (A3) was used.

The surface integral can be transformed into a line integral by using the Green formulas[10]

$$\int_G\int \frac{\partial f}{\partial \bar z}\,dx\,dy = \frac{1}{2}\int_G\int\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)dx\,dy = \frac{1}{2}\int f\,dy - \frac{i}{2}\int f\,dx$$
$$= \frac{1}{2i}\int f\,dx + \frac{1}{2}\int f\,dy = \frac{1}{2i}\int_{\mathcal{C}} f(z)\,dz \tag{32}$$

and then we obtain from (31)

$$\alpha\frac{\nu\nu'}{2\pi i}\int_{\mathcal{C}}(1+\alpha\gamma'')\frac{1}{z-z''}\,\frac{1}{z'-z''}\,dz''$$
$$= -\alpha\nu\nu'\frac{1}{2\pi i}\cdot 2\pi i\frac{1}{\alpha}\left[\left.\frac{\partial\gamma}{\partial z}\right|_{z=z_0}\right]^{-1}\frac{1}{z-z_0}\,\frac{1}{z'-z_0}$$
$$= -\nu\nu'\left(\left.\frac{\partial\gamma}{\partial z}\right|_{z=z_0}\right)^{-1}\frac{1}{z-z_0}\cdot\frac{1}{z'-z_0} \tag{33}$$

which is nothing but the negative of the kernel of the projector $P_k = f_k(\tilde{f}_k,\cdot)$ on the eigenvector f_k corresponding to the eigenvalue $\lambda_k \equiv z_0$ of B_k. The minus comes from the fact that the sense of integration on $\mathcal{C}$ must be counterclockwise for the continuous spectrum, and therefore it is clockwise for the region containing the pole.

In deducing (33) we took into account that the contribution of the half-circle at infinity in the upper-half-plane is obviously null.

This ends the proof for the existence of the wave operators with desired properties for $|k| \neq k^*$. If $k = k^*$ (eigenvalue "*in statu nascendi*" at the edge λ^* of the essential spectrum) no such bounded operators can be found. This situation is always encountered in the infinite or semi-infinite medium case ($k \in \mathbb{R}$), but only for exceptional values of the slab widths in the periodic case [$k \in \mathbb{Z}$; see Ref. 1].

We have then the main result:

Theorem. For every $\delta > 0$ and every Borel set $\Delta \in \mathbb{C}$ at nonzero distance from $-\lambda^*$, the subspace

$$\mathcal{H}_{\delta,\Delta} = \int_{\substack{\oplus \\ |k\pm k^*|>\delta}} \mathcal{L}_k\,dk \oplus \int_{\substack{\oplus \\ |k\pm k^*|\leqslant\delta}} E_k(\Delta)\mathcal{L}_k\,dk \tag{34}$$

is invariant under the operator A [E_k is given by (10)] and A restricted to $\mathcal{H}_{\delta,\Delta}$ is similar to a normal operator.

4. REMARKS AND CONCLUSIONS

1. For slab and parallelepipedal geometries with periodic or specularly reflecting boundary conditions the theorem can be strengthened in the sense that with the exception of a denumerable set of critical dimensions of the bodies, the operator A is similar to a normal operator. For those critical values, when it so happens that λ^* is also a discrete eigenvalue, we can obtain only a weaker result, similar to (34) (see Ref. 1).

2. The proof is valid also for other models, for instance one- and three-dimensional energy-dependent models with constant collision frequency (the spectrum of B_k is then reduced to a line parallel to the real axis plus, possibly, a discrete eigenvalue), or one-dimensional energy-dependent models with energy-dependent collision frequency (the spectrum of B_k is then only the curve $\mathcal{C}$ given by (11) and, possibly, a discrete eigenvalue). As mentioned in the Introduction, these models are related to the BGK model. Their essential feature is that the continuous spectrum of B_k does not cover an area in the complex plane, but is reduced to a one-dimensional curve. Then, the singularity introduced by the factor $1/(z - z')$ in (19) is stronger than in the case considered here and it must be treated by means of the approximate kernels $G_\epsilon = (z' - z + i\epsilon)^{-1}$ exactly as in Ref. 1. Moreover, the verification of the relations (a′), (a″) does not appeal to the theory of generalized analytic functions, as from the beginning we are dealing with true analytic functions of the appropriate variable z and the involved integrals are already line integrals.

3. The connection of the wave operators with the generalized eigenfunctions of A and the properties of "completeness" and "orthogonality" can be discussed exactly as in Ref. 1.

4. The proof given here is constructive and takes advantage of some particular aspects of the models and of the setting, especially of the facts that the perturbation J is self-adjoint and of rank 1 and the Fourier transformation diagonalizes the operator A.

5. The further generalizations of the result should deal with perturbations and boundary conditions for which only an existence theorem of the wave operators is implied.

ACKNOWLEDGMENTS

The kind hospitality of the Mathematical Department of the University Kaiserslautern, where this work was initiated, is warmly acknowledged. One of the authors (V.P.) thanks the CNR for financial support and is grateful for the pleasant stays at the Institute of Physics of the University of Milano and at the Institute of Mathematics of the Polytechnic Institute of Milano, where this paper was completed.

APPENDIX

In the main part of this paper we have used the notation and some results of the theory of generalized analytic functions.[10] The main tool is the following result, also known as Poincaré's lemma. If h is an integrable function of x and y in a closed region G of the plane and if we denote $x + iy$ by z, then

$$f(z') = -\frac{1}{\pi}\int_G\int \frac{h(z)}{z - z'}\, dx\, dy \tag{A1}$$

is differentiable a.e. and

$$\frac{\partial f}{\partial \bar{z}'} = h(z') \tag{A2}$$

This means that, in the sense of distributions,

$$\frac{\partial}{\partial \bar{z}'}\frac{1}{z - z'} = -\pi\delta(x - x')\delta(y - y') = -\pi\delta(z - z') \tag{A3}$$

A formal proof of this result is very easy (see, e.g., Ref. 3).

Equation (26) of the main text can be rewritten as follows:

$$\gamma(z') = \int\int \frac{\nu^2(z)\, dx\, dy}{(z - z')\rho(z)} \tag{A4}$$

where

$$\rho(z) = \frac{\partial(x, y)}{\partial(v, \mu)} \tag{A5}$$

is the Jacobian of (x, y) with respect to (v, μ).

Accordingly,

$$\frac{\partial \gamma}{\partial \bar{z}'} = -\pi\nu^2(z')/\rho(z') \tag{A6}$$

This result can be used to express ν''^2 in Eq. (30) in order to obtain Eq. (31). In addition Eq. (A3) is used in Eq. (31). Equation (32) is nothing but the complex version of the Green formulas.[10]

We remark that the results presented in this Appendix are also valid if the domain G is unbounded as required in the main text, although this case does not seem to have been considered in the literature. To prove it, it suffices to assume that $h(z)/(z - z')$ is integrable for any z' in G. [This is certainly the case of ν^2/ρ provided $\xi(v)$ exists almost everywhere, is bounded for finite v's, and does not grow too fast for $v \to \infty$]. Then for any fixed z' we can split G into G', bounded and containing z' and its complement in G, G''. The integration of $h(z)/(z - z')$ over G'' yields a holomorphic function of z', having zero derivative with respect to $\bar{z}'$; then the usual lemma can be applied to G', in order to get Eq. (A2).

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